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Dynamics of spiral waves in non-equilibrium media

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Abstract. A class of non-equilibrium media described by equations close to gradient one is considered. For the analysis of the field structure dynamics in such media an asymptotic method is proposed where the generating solution is that of the gradient system. The analysis is based on the generalised Ginzburg-Landau equation. For $\varepsilon = 0$ this equation can be written as $\partial a/\partial t = -\delta F[a]/\delta a^*$ where F is the Lyapunov functional:

$$F = \int [f(|a|^2) + |\nabla a|^2]\, dx\, dy$$

and solutions are possible in the form of static spiral waves centred on the point (x, y). When $0 < \varepsilon \ll 1$ a solution is sought in the form of an asymptotic series with the first term having the form of a spiral wave, but the parameters x, y and phase will be slow functions of time. Using this method the interaction of a pair of spiral waves in media with hard and soft excitation is investigated and the stochastic drift of a spiral wave in a periodically inhomogeous field is predicted.

1. Introduction

Most simple and natural mechanisms for the origin of spatial disorder in homogeneous isotropic nonlinear media are, at first sight, similar to those revealed in the onset of spatial chaos in Hamiltonian systems and similar mechanisms. This is readily illustrated by a trivial spatio-temporal analogy for homogeneous media described by gradient equations. For example, under the action of a static external field periodically inhomogenous in space (inhomogeneous heating) a periodic chain of convective rolls may change over to disordered state that will look, for example, like a chaotic sequence of rolls rotating in opposite directions (Love *et al* 1983). The mechanism of the onset of deterministic chaos in this case is analogous to the known mechanism of the stochastisation of a nonlinear oscillator under the action of a harmonic force. However, such a static spatial chaos is relevant only for unbounded media.

In bounded regions, static disorder cannot, evidently, evolve: any fields that seem to be disordered at finite distances may be periodically extended in space.

Nevertheless, spatial disorder may exist in bounded systems, but it is time dependent rather than static. An example of such a spatio-temporal chaos is a disordered motion of spiral waves that we have found in static periodically inhomogeneous fields. In a more general case, such a random walk of structures (spirals) may be caused by the interaction of the structures rather than by regular inhomogeneities (see, for example, Aranson *et al* 1987, Linde 1984). The random walk of the field structures resulting in the mixing of their orbits even in limited regions of space seems to be one of the most general mechanisms of the self-generation of spatio-temporal deterministic chaos—dynamical turbulence—in nonlinear non-equilibrium media.

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The character of nonlinear processes (the formation and interaction of structures, the evolution of turbulence, and so on) in non-equilibrium media depends significantly on the character of transient processes that are implemented behind a critical point. It is most important here to distinguish between oscillatory and aperiodic instabilities. The aperiodic behaviour generally results in the formation (as $t \to \infty$) of various spatial structures.

The stability of such structures near the critical point for the case under study indicates that they correspond to a minimum of some functional. The process of pattern formation and, consequently, the dynamics of media with such behaviour can, as a rule, be described by a gradient model (Landau and Lifshitz 1982)

$$\frac{\partial a}{\partial t} = -\frac{\delta F[a]}{\delta a^*} \tag{1}$$

where $F[a]$ is the Lyapunov functional for which we use the 'free energy' functional. Here a may have the sense of a 'complex order parameter'.

For the oscillatory 'transient behaviour' of the medium, turbulence may occur. Formally, this situation corresponds to the appearance of a non-trivial attractor (e.g. attracting stochastic set) in the functional space. Such a behaviour of nonlinear non-equilibrium media cannot be described by a gradient model. Because the terms responsible for the deviation of the system from a gradient model are usually obtained as a result of the expansion in the supercriticality parameter, for moderately high supercriticalities it seems practical to consider a special class of models that are close to gradient models:

$$\frac{\partial a}{\partial t} = -\frac{\delta F}{\delta a^*} - \varepsilon \tilde{\phi}(a, x, y, t). \tag{2}$$

Here $\tilde{\phi}$ is the operator that may depend on the dynamic variable as well as on the x, y coordinates and time t, ε is the small parameter of the problem.

Indeed, for $\varepsilon = 0$, the initial model has a solution in the form of stable structures and, although for $0 < \varepsilon \ll 1$ these structures are no longer static, they can be used as a generating solution in the construction of a perturbation theory for the description of a non-trivial behaviour of the non-equilibrium media under study.

Consider as an example the media described by a generalised two-dimensional Ginzburg–Landau (GL) equation

$$\frac{\partial a}{\partial t} = af(|a|^2) + \Delta a + i\varepsilon(G(|a|^2) + c\Delta a + \ldots). \tag{3}$$

It is, apparently, represented in the form (2) with the functional

$$F = \int \{\tilde{f}(|a|^2) + |\nabla a|^2\}\, dx\, dy \qquad \frac{\partial}{\partial a^*}\tilde{f}(|a|^2) = -af(|a|^2). \tag{4}$$

Below we shall investigate the dynamic solution of (3) for which the generating solution has the forms of the spiral wave 'origin' (Hagan 1982) (ρ, θ are polar coordinates)

$$a^0 = \phi^{(0)}(\rho)\, e^{\pm im\theta} \qquad \rho = \sqrt{x^2 + y^2} \qquad \theta = \tan^{-1}\frac{y}{x} \qquad m = 1, 2, \ldots \tag{5}$$

that are most typical of (3) when $\varepsilon = 0$. With the perturbation ($\varepsilon \neq 0$) taken into account, the generating solution is distorted. However, the smallness of ε permits us to seek a new solution asymptotically. In the general case, the perturbation has two consequences: the spiral rotation and the drift of the spiral core. The rotation is accompanied by the whirling of the initial (generating) wavefront and the solution takes on a form

$$a=\phi^{(0)}(\rho)\exp[\mathrm{i}(m\theta+\Omega t-\psi(\rho))] \tag{6}$$

where $\psi(0)=0$ and $\psi(\rho)\to k\rho$ as $\rho\to\infty$ (k is the asymptotic wavenumber, Ω is the spiral rotation frequency). This situation occurs, in particular, when the weak complexity of (3) $G(|a|^2)\neq 0$ is taken into account.

When we are concerned with the perturbations that cause the drift of the spiral core, the proximity of the perturbed system to a gradient one guarantees a low-velocity drift.

The paper consists of seven sections. Some details of the asymptotic method for spirals are given in section 2. The motion of a localised spiral in the framework of two versions of the GL equation is considered in section 3. Also, some analytical and numerical results, concerning the problem of the spiral wave stability, are represented. In section 4 pair interactions of the spiral waves are investigated. Section 5 is devoted to the problem of the non-localised spiral wave dynamics in the media with soft excitation. In section 6, the equations of motion of the non-localised spiral pair are obtained by means of the spiral wave-disclination analogy. In the discussion the connection with spiral wave dynamics in excitable media is considered.

2. Asymptotic method

We shall consider the evolution of those solutions of (3) that are stable (or long-lived) in time for $\varepsilon=0$. It is known that such solutions are, as a rule, simple spiral waves with $m=\pm 1$ (Hagan 1982). Therefore, below we shall restrict ourselves to the analysis of these waves, although an asymptotic method can also be used for the case $|m|\neq 1$, provided that the corresponding solutions are stable.

Consider now for $\varepsilon=0$ solution (5) in the form

$$a^{(0)}(x,y,t)=\phi^{(0)}(\sqrt{(x-x_0)^2+(y-y_0)^2})\exp\left[\mathrm{i}\left(\tan^{-1}\frac{y-y_0}{x-x_0}-\varphi_0\right)\right] \tag{7}$$

where x_0 and y_0 are the coordinates of the spiral wave core and φ_0 is the phase characterising, for example, the position of the 'leading' front (the phase can be chosen up to a constant). We assume that for $\varepsilon\neq 0$, the values x_0, y_0 and φ_0 are slow functions of time and seek a solution $a_\varepsilon(x,y,t)$ in the form of

$$a_\varepsilon(x,y,t)=\phi^{(0)}(\sqrt{(x-x_0(t))^2+(y-y_0(t))^2})\times\exp\left\{\mathrm{i}\left[\tan^{-1}\left(\frac{y-y_0(t)}{x-x_0(t)}\right)-\varphi_0(t)\right]\right\}+\sum\varepsilon^n a_n. \tag{8}$$

Substituting (7) into (3) for the functions u_n and v_n, we obtain a system of linear differential equations ($a_n=u_n+\mathrm{i}v_n$):

$$\begin{aligned}&\Delta u_n+(f(|a^0|^2))+u_0f'_{u_0}(|a^0|^2))u_n+u_0f'_{v_0}(|a^0|^2)v_n=H_\mathrm{R}^{(n)}\\&\Delta v_n+(f(|a^{(0)}|^2)+v_0f'_{v_0}(|a^0|^2))v_n+v_0f'_{u_0}(|a^0|^2)u_n=H_\mathrm{I}^{(n)}\end{aligned} \tag{9}$$

where $H^{(n)}=H_{\mathrm{R}}^{(n)}+\mathrm{i}H_{\mathrm{I}}^{(n)}$ are the expressions including corrections from the previous approximations and omitting u_n and v_n. For system (9) to be solved, the right-hand side must be orthogonal to the core of the conjugate system (Gershkov and Ostrovsky 1981, Aranson *et al* 1984). It can be easily proved that (9) is a self-conjugate system for a class of functions u_n, v_n meeting the boundary conditions: $u_n(\rho)$, $v_n(\rho)\to 0$ when $\rho\to\infty$ and $|u_n|$, $|v_n|<0$ when $\rho\to 0$. Direct substitution shows that the functions

$$a_{\theta_0}^{(0)}=\mathrm{i}a^{(0)}=\mathrm{i}\phi^{(0)}\mathrm{e}^{\mathrm{i}(\theta_0-\varphi_0)}$$
$$a_x^{(0)}=\left(\phi_\rho^{(0)}\cos(\theta_0-\varphi_0)-\mathrm{i}\frac{\sin(\theta_0-\varphi_0)}{\rho}\phi^{(0)}\right)\mathrm{e}^{\mathrm{i}(\theta_0-\varphi_0)} \tag{10}$$
$$a_y^{(0)}=\left(\phi_\rho^{(0)}\sin(\theta_0-\varphi_0)+\mathrm{i}\frac{\cos(\theta_0-\varphi_0)}{\rho}\phi^{(0)}\right)\mathrm{e}^{\mathrm{i}(\theta_0-\varphi_0)}$$

satisfy the homogeneous system (9), i.e. they are eigenfunctions of the core. Then solvability conditions can be written in the form

$$\mathrm{Re}\int[H^{(n)}a_x^{(0)*}]\,\mathrm{d}x\,\mathrm{d}y=0 \qquad \mathrm{Re}\int[H^{(n)}a_y^{(0)*}]\,\mathrm{d}x\,\mathrm{d}y=0$$
$$\mathrm{Im}\int[H^{(n)}a^{(0)*}]\,\mathrm{d}x\,\mathrm{d}y=0. \tag{11}$$

Most interesting are equations to the first approximation. In this case, the correction $H^{(1)}$ can be represented in the form

$$H^{(1)}=-\frac{\partial a^{(0)}}{\partial t}-\tilde{\phi}[a^{(0)},\varepsilon x,\varepsilon y,\varepsilon t]. \tag{12}$$

Substituting (12) into (11), we shall obtain the 'equation of motion' of the spiral core (taking into account $\partial a^{(0)}/\partial t=-\dot{x}_0a_x^{(0)}-\dot{y}_0a_y^{(0)}-\mathrm{i}\dot{\varphi}_0a^{(0)}$)

$$\int\mathrm{d}x\,\mathrm{d}y(V\nabla_0a^{(0)})\nabla a^{(0)*}=-\mathrm{Re}\int\tilde{\phi}\nabla_0a^{(0)*}\,\mathrm{d}x\,\mathrm{d}y \tag{13}$$

where

$$V=(\dot{x}_0,\dot{y}_0,\dot{\varphi}_0) \qquad \nabla_0=\left(\frac{\partial}{\partial x_0},\frac{\partial}{\partial y_0},\frac{\partial}{\partial\varphi_0}\right)$$

or

$$\dot{x}_0\int|a_x^{(0)}|^2\,\mathrm{d}x\,\mathrm{d}y=\mathrm{Re}\int\tilde{\phi}a_x^{(0)*}\,\mathrm{d}x\,\mathrm{d}y$$
$$\dot{y}_0\int|a_y^{(0)}|^2\,\mathrm{d}x\,\mathrm{d}y=\mathrm{Re}\int\tilde{\phi}a_y^{(0)*}\,\mathrm{d}x\,\mathrm{d}y \tag{14}$$
$$\dot{\varphi}_0\int|a^{(0)}|^2\,\mathrm{d}x\,\mathrm{d}y=\mathrm{Im}\int\tilde{\phi}a^{(0)*}\,\mathrm{d}x\,\mathrm{d}y$$

(the values of the variables εx and εy in the expression for $\tilde{\phi}(\varepsilon x,\varepsilon y)$ are taken for the spiral core). It is seen from (12) and (14) that the equations of the spiral wave motion are essentially non-Newtonian: velocity, rather than acceleration, is proportional to the applied force. Apparently, this is stipulated by a gradient form of the generating system (3). Corrections to the following approximations can be determined recurrently.

3. Spiral wave motion in the external fields in media with hard excitations

We shall derive equations of motion for the spiral wave core, but first we shall consider an important circumstance. The solvability conditions (11) or (14) can be met correctly only if the functions $a^{(0)}$, $a_x^{(0)}$ and $a_y^{(0)}$ vanish rather fast as $\rho \to \infty$, i.e. the generating solution (5) must be localised: $\phi^{(0)}(\rho) \to 0$ as $\rho \to \infty$.

Possible existence of a spiral wave localised in space was first reported by Petviashvili and Sergeev (1984). In that work, for a two-dimensional GL equation with hard exitation and long-wave instaiblity,

$$\frac{\partial a}{\partial t} = -a + (\beta + \mathrm{i}\beta')|a|^2 a - (1 + \mathrm{i}\gamma)|a|^4 a + (1 - \mathrm{i}c)\Delta a \qquad \beta, \beta', \gamma, c = \text{constant} \tag{15}$$

the authors found a stationary axially symmetric solution in the form of a rotating spiral wave (6). However, in that paper nothing was said about the stability of such solutions.

System (15) has stationary localised solutions only when $\beta > 2$. The dynamics of the solutions of system (15) for $\beta' = \gamma = c = 0$ can be investigated by analysing the 'free energy functional'

$$F = \int \left(|a|^2 - \frac{\beta |a|^4}{2} + \frac{|a|^6}{3} + |\nabla a|^2 \right) \mathrm{d}x\, \mathrm{d}y.$$

It can be represented in the form

$$F = \int \{ |a|^2 (1 - |a|^2/\sqrt{3})^2 + |\nabla a|^2 + (2/\sqrt{3} - \beta/2)|a|^4 \}\, \mathrm{d}x\, \mathrm{d}y.$$

Apparently, for $\beta = 4/\sqrt{3} = 2.3094\ldots$ the functional F is non-negative for any localised (in the sense of finiteness of the value of F) initial perturbation. The first derivatives of F with respect to time are equal to zero only in stationary solutions of (15) and, therefore for $\beta \leq 2.3094\ldots$ any localised perturbations collapse as $t \to \infty$.

On the other hand, for $\beta > 2.3094\ldots$ the analysis shows that in the general case it is 'energetically profitable' for the localised perturbation to spread (and for $\beta \leq 2.3094\ldots$ to collapse). In this case, travelling transfer fronts like those observed in excitable media were realised in the system. Indeed, consider a solution in the form of a cylindrical front

$$|a|^2 = \begin{cases} |a_{\mathrm{m}}|^2 = \dfrac{\beta}{2} + \sqrt{\dfrac{\beta^2}{4} - 1} & \rho \leq \rho_0,\ \rho_0 \gg 1 \\ \sim |a_{\mathrm{m}}|^2 \exp[2(\rho_0 - \rho)] & \rho > \rho_0 \end{cases}$$

where $|a_{\mathrm{m}}|^2$ is a maximum possible stationary value of intensity $|a_{\mathrm{m}}|^2$. For $\rho_0 \gg 1$, the contribution from the exponential 'tail' (when $\rho > \rho_0$) is small in comparison with the contribution from the region $0 \leq \rho \leq \rho_0$, which is equal to

$$F \approx \pi\rho_0^2 \left(|a_{\mathrm{m}}|^2 - \frac{\beta |a_{\mathrm{m}}|^4}{2} + \frac{|a_{\mathrm{m}}|^6}{3} \right) = \pi\rho_0^2 \left(\frac{1}{2} - \frac{|a_{\mathrm{m}}|^4}{6} \right).$$

For $|a_{\mathrm{m}}|^2 > \sqrt{3}$ (which corresponds to $\beta = 2.3094\ldots$) the spreading of the solution (i.e.

the increase of ρ_0) corresponds to a decrease of F, i.e. it is 'energetically profitable'. This situation describes the propagation of a cylindric front. The same considerations are valid for a three-dimensional case.

For a more detailed analys, we carried out a direct numerical integration of (15) for $\beta' = \gamma = c = 0$. For a spiral wave it is easy to show that $\Omega = \psi(\rho) = 0$, i.e. the solution is written in the form (5). Besides, the gradient form allows for a significant simplification of its numerical study (in particular, it guarantees a trivial (oscillation-free) temporal behaviour of $a(x, y, t)$. In our computer experiments the integration was accomplished by an implicit split-step scheme accurate to second order (Aranson *et al* 1989); the Laplacian operator on the right-hand side was calculated by a FFT method. The number of harmonics was chosen to be 64×64.

In the numerical simulation with single-armed spirals ($m = \pm 1$) the following phenomenon was observed: quite quickly (about 1-2 characteristic times) a rather general initial condition generated a spiral solution similar to that depicted in figure 1(a). Then, very slowly, the spiral wave either spread and converted into an ordinary non-localised spiral (see, for example, Malomed and Rudenko 1988) thus entering a limit state, or it collapsed. Thus, localised spiral solutions are metastable. However, their lifetime may be arbitrary large as $\beta \rightarrow \beta_c = 4/\sqrt{3}$, because the velocity of spreading is $\sim|\beta - \beta_c|$. At the same time, the spiral solutions with $|m| = 2$ break down in 1-2 characteristic units of time.

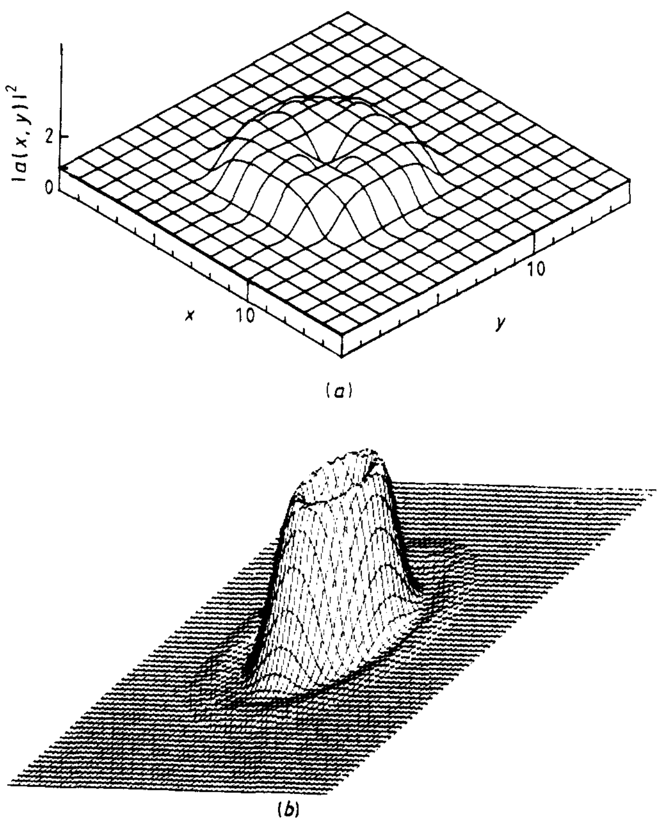


Figure 1. Distribution of $|a(x, y)|^2$ for the localised solution (5) (a) for (15) with $\beta = 2.5$ and $\gamma = c = 0$, (b) for (16) with $\beta = 2.5$, $k_0^2 = 1$. The size of the region of integration is 20×20.

It should be emphasised that even with the spiral spread, the core structure does not change. Thus the construction of the equations of motion for the spiral core does not depend significantly on its slow spreading, at least until the spiral 'diameter' is of order $1/\varepsilon$. This condition may be met if β is chosen to be close to β_c.

We found that the stable spiral waves exist in the short-wave GL equation with hard excitation

$$a_t = -a + \beta|a|^2 a - |a|^4 a - (k_0^2 + \Delta)^2 a \qquad \beta, k_0^2 = \text{constant}. \tag{16}$$

This equation is a generalisation of the well known Swift–Hohenberg equation to the case of an oscillatory instability (Swift and Hohenberg 1977). Here k_0^2 represents the scale of the initial short-wave instability. This situation occurs in binary liquid convection (Moses and Steinberg 1986, Rehgberg *et al* 1988, Kolodner and Surko 1988). The chaos of spiral waves in binary liquid convection was experimentally observed by Linde (1984).

The stable spiral solution of (16) is shown in figure 1(*b*). It is difficult to prove the stability analytically, but some considerations can be inferred from the free energy functional

$$F = \int \left(|a|^2 - \frac{\beta |a|^4}{2} + \frac{|a|^6}{3} + |(k_0^2 + \Delta) a|^2 \right) \mathrm{d}x\, \mathrm{d}y. \tag{17}$$

By analogy with the above considerations, we can obtain that in the range $4/\sqrt{3} \leqslant \beta \leqslant (4/\sqrt{3})(k_0^2+1)^{1/2}$ the collapse or spreading of localised solutions cannot occur. Apparently, the propagation of the front is not profitable in the model considered because of the differential operator structure. In contrast, strongly inhomogeneous distributions, like those shown in figure 1(*b*), must minimise the functional (18).

Consider now complete equations (15) or (16), taking into account the complexity of the coefficients (for definiteness, $\beta \to \beta + \mathrm{i}\beta'$) and the presence of a weak parametric effect at a frequency close to the natural frequency of self-oscillations. Then, the perturbed equation will be written in the following form:

$$\frac{\partial a}{\partial t} = -a + (\beta + \mathrm{i}\beta')|a|^2 a - |a|^4 a - (k_0^2 + \Delta)^2 a + \varepsilon \chi(|a|^2) \tilde{f}(\varepsilon x, \varepsilon y)\, \mathrm{e}^{-\mathrm{i}\delta t}. \tag{18}$$

Here δ is a small (of order ε) detuning and the function $\chi(|a|^2)$ describes the nonlinear response of the medium to the external effect whose spatial distribution is described by the function $\tilde{f}(\varepsilon x, \varepsilon y)$, which is smooth in comparison with the size of the spiral. The function $\chi(|a|^2)$ can be represented as a series in $\chi(|a|^2) = A_0 + A_1|a|^2 + A_2|a|^4 + \ldots$.

The term A_0 does not make a contribution because it is orthogonal to all eigenfunctions of the core of the conjugate system. Therefore, we shall assume for simplicity that $\chi(|a|^2) \sim |a|^2$, i.e. we shall restrict ourselves to the first non-trivial term in the series (the coefficient A_1 can be assumed to be equal to unity).

Perturbations may not only cause the drift of the spiral core but also initiate small-amplitude large-scale perturbations that can be interpreted as the radiation fields (Gorshkov and Ostrovsky 1981). For the case under study, the radiation fields are not essential, since the waves radiated by the spiral quickly damp as they move away from the centre of the spiral (the damping constant of plane waves in (15) is approximately $\geqslant 1$). Therefore we can restrict our attention to the motion of the spiral core. The correction $H^{(1)}$ in the case of interest has the form

$$H^{(1)} = \dot{x}_0 \frac{\partial a^{(0)}}{\partial x} + \dot{y}_0 \frac{\partial a^{(0)}}{\partial y} + \mathrm{i}\dot{\varphi}_0 a^{(0)} + \mathrm{i}\beta' |a^{(0)}|^2 a^{(0)} + \varepsilon |a^{(0)}|^2\, \mathrm{e}^{-\mathrm{i}\delta t} \tilde{f}(\varepsilon x, \varepsilon y). \tag{19}$$

Using the orthogonality condition (11) we shall obtain the following system of equations for the velocities of the spiral core:

$$\begin{aligned}\dot{x}_0 &= B\,\mathrm{Re}[\tilde{f}(\varepsilon x_0, \varepsilon y_0)\,\mathrm{e}^{\mathrm{i}(\varphi_0-\delta t)}]\\ \dot{y}_0 &= B\,\mathrm{Im}[\tilde{f}(\varepsilon x_0, \varepsilon y_0)\mathrm{e}^{\mathrm{i}(\varphi_0-\delta t)}]\\ \dot{\varphi}_0 &= \tilde{\Omega}\end{aligned} \qquad (20)$$

where

$$B = \tfrac{1}{3}\varepsilon\left[\int_0^\infty (\phi^{(0)}(\rho))^3\rho\,\mathrm{d}\rho\left(\int_0^\infty \mathrm{d}\rho\rho((\phi_\rho^{(0)}(\rho))^2 + \frac{1}{\rho^2}(\phi_{(\rho)}^{(0)})^2\right)^{-1}\right] = \text{constant}$$

$$\tilde{\Omega} = \beta' \int_0^\infty \rho(\phi^{(0)}(\rho))^4\,\mathrm{d}\rho\left(\int_0^\infty \rho(\phi^{(0)}(\rho))^2\,\mathrm{d}\rho\right)^{-1} = \text{constant}.$$

In a simple case when $\tilde{f}(x, y) = \text{constant} = C_0$, (20) can be represented in the form

$$\dot{x}_0 = C_0 B\cos(\varphi_0 - \delta t) \qquad \dot{y}_0 = C_0 B\sin(\varphi_0 - \delta t) \qquad \dot{\varphi}_0 = \tilde{\Omega}$$

or

$$\dot{x}_0 = C_0 B\cos((\tilde{\Omega} - \delta)t + \psi_0) \qquad \dot{y}_0 = C_0 B\sin((\tilde{\Omega} - \delta)t + \psi_0)$$

where $\psi_0 = \varphi_0(t = 0)$ characterises the phase of the leading front of the spiral at the initial moment of time. Apparently, if $\tilde{\Omega} \neq \delta$, the spiral rotates circularly with a radius $|C_0B|$ around the core that is determined by the values of $x_0(0)$ and $y_0(0)$. For $\tilde{\Omega} = \delta$, the spiral wave drifts with a constant velocity equal to $|C_0B|$ at an angle ψ_0 to the axis x†. According to the terminology adopted by Agladze *et al* (1987) this phenomenon is called spiral wave resonance. Let us now represent the function $\tilde{f}(\varepsilon x, \varepsilon y)$ in the form $\tilde{f}(\varepsilon x, \varepsilon y) = \cos kx + \mathrm{i}\sin ky$ and analyse the situation when there is no spiral rotation and detuning: $\tilde{\Omega} = \delta = 0$. Then (20) can be represented in the form of a Hamiltonian system with a Hamiltonian:

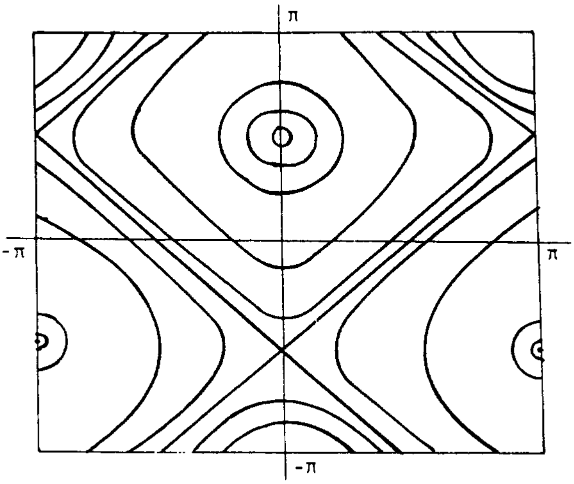
$$H = \frac{B}{k}(\sin ky_0 + \cos kx_0) \qquad \dot{x}_0 = B\cos ky_0 = \frac{\partial H}{\partial y_0} \qquad \dot{y}_0 = B\sin kx_0 = -\frac{\partial H}{\partial x_0}. \qquad (21)$$

The phase portrait of (21) is shown in figure 2(*a*). One can see that system (21) has a separatrix network covering the whole x, y plane. With weak non-stationary perturbations, the separatrix network will apparently break up and a stochastic spider web analogous to that considered in Chernikov *et al* (1987) will appear. The stochastic spider web (18) indicates that the spiral wave may stochastically drift along the x, y plane over arbitrary long distances. Allowance for the spiral rotation $\tilde{\Omega} \neq 0$ (or detuning) yields non-stationary (time-periodic) perturbations. In this case we shall have equations for the coordinates of the spiral core

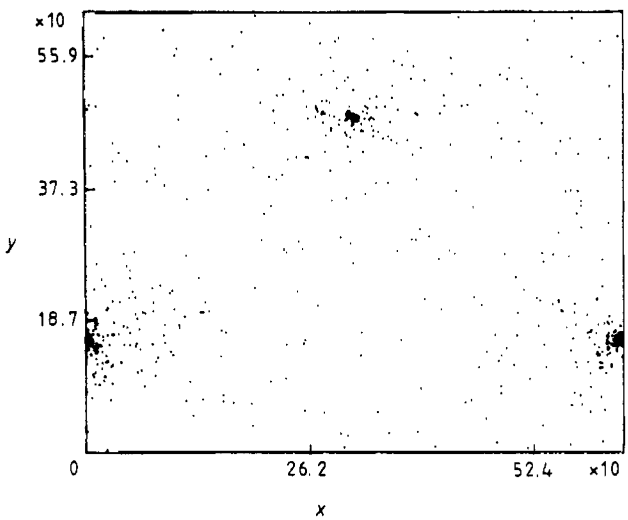
$$\begin{aligned}\dot{x}_0 &= B(\cos\tilde{\Omega}t\cos ky_0 - \sin\tilde{\Omega}t\sin kx_0)\\ \dot{y}_0 &= B(\cos\tilde{\Omega}t\sin kx_0 + \sin\tilde{\Omega}t\cos ky_0)\end{aligned} \qquad (22)$$

that explicitly depend on time, which provides for the breaking of the separatrix network in the unperturbed system (22) (see figure 2(*b*)).

† In the general case, however, the spiral cannot drift over infinitely long distances; with the corrections to the following approximations taken into account, the frequency of the spiral rotation $\tilde{\Omega}$ will change periodically, which will result in the change of the drift direction.



(*a*)



(*b*)

Figure 2. (*a*) Structure of the phase plane of (19) for $H = (B/k)(\sin ky + \cos kx)$. (*b*) Poincaré section by the period $2\pi/\tilde{\Omega}$ in (20) for $\tilde{\Omega} = k = 1$, $B = 3$.

The motion of the spiral core is described by quite general third-order equations for which chaotic solutions are known to be rather typical; therefore the stochastic spiral drift in external fields must be a fairly common occurrence.

4. Dynamics of spiral pairs

We shall describe two cases of spiral pair interaction: (i) the interaction of spirals rotating in the same direction, i.e. solutions of the form (5) having the same sign before θ_0 (it is natural to call them like spiral pairs) and (ii) the interaction of unlike spiral pairs, i.e. spirals rotating in opposite directions (those with different signs before θ_0). This problem is meaningful, in particular, when considering the stability of a toroidal vortex (figure 3). If the radius of the vortex ring R is much larger than the characteristic transverse size of the vortex, the vortex curvature may be neglected. Then, in the plane crossing the rotation axis of the vortex the field structure is specified by a pair of spirals rotating in opposite directions (figure 3(*b*)). As a result of spiral interaction the vortex radius R will change in time. The law of the variation was estimated by Panfilov *et al* (1986) and Brazhnik *et al* (1987):

$$\dot{R} = -\frac{\nu}{R} \qquad \nu = \text{constant}. \tag{23}$$

These estimates show that the toroidal vortex collapses in time. However, in experiments stable toroidal vortices were also observed (Brazhnik *et al* 1987). We believe that a stable 'bound state' of a spiral pair, i.e. the state where the distance between the spirals does not change in time, must correspond to a stable toroidal vortex.

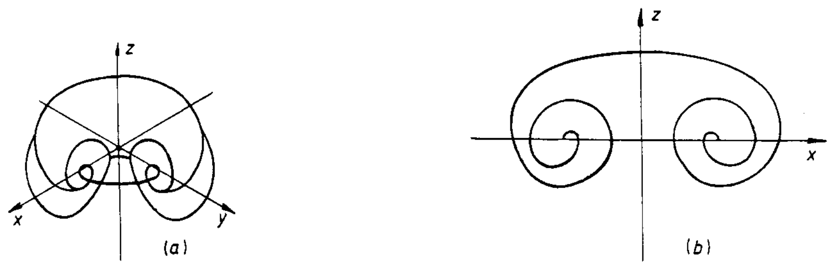


Figure 3. (*a*) Structure of the toroidal vortex ring; (*b*) cross section of vortex ring for $R \gg 1$.

Consider first the interaction of widely spaced spirals. Since the field of each spiral decreases exponentially fast from the core (one can easily obtain the asymptotic form), a widely spaced (at a distance much larger than the characteristic size of the spiral) spiral pair has a field that is close to the superposition of the fields of each spiral, i.e.

$$a = \phi^{(0)}(\rho_1)\exp[\mathrm{i}(\theta_1 - \varphi_1)] + \phi^{(0)}(\rho_2)\exp[\mathrm{i}(\theta_2 m - \varphi_2)] \tag{24}$$

where

$$\rho_{1,2} = [(x - x_{1,2})^2 + (y - y_{1,2})^2]^{1/2} \qquad \theta_{1,2} = \tan^{-1}\left(\frac{y - y_{1,2}}{x - x_{1,2}}\right).$$

Like spirals (i.e. rotating in the same direction) correspond to $m = 1$, while unlike spirals (rotating in opposite directions) correspond to $m = -1$. In this case, the field of one spiral that slightly perturbs the other one causes the drift.

Substitute (24) into (15) or (16) and take into account the nonlinear terms responsible for the spiral interaction. For definiteness, we shall speak of the solution with the index 1 (the same consideration holds for the other spiral wave). The correction $H_1^{(1)}$ for the chosen solution has the following form (the field of the second spiral at the core of the first one will be the perturbation):

$$H_1^{(1)} = -\frac{\partial a^{(0)}}{\partial t} + \tilde{\phi}$$

$$\tilde{\phi} = (\beta|a_1^{(0)} + a_2^{(0)}|^2 - |a_1^{(0)} + a_2^{(0)}|^4)(a_1^{(0)} + a_2^{(0)}) - \beta(|a_1^{(0)}|^2 a_1 + |a_2^{(0)}|^2 a_2) + |a_1^{(0)}|^4 a_1 + |a_2^{(0)}|^4 a_2^{(0)}. \tag{25}$$

Similarly

$$H_2^{(1)}=\frac{\partial a_2^{(0)}}{\partial t}+\tilde{\phi}.$$

The orthogonality conditions (13) and (14) may, apparently, be represented in a vector form

$$(\hat{m}V_{1,2})=-\mathrm{Re}\int \nabla_{1,2}a_{1,2}^{(0)*}\tilde{\phi}\,\mathrm{d}x\,\mathrm{d}y \tag{26}$$

where $V_{1,2}=(\dot{x}_{1,2},\dot{y}_{1,2},\dot{\varphi}_{1,2})$ is the velocity vector,

$$\nabla_{1,2}=\left(\frac{\partial}{\partial x_{1,2}},\frac{\partial}{\partial y_{1,2}},\frac{\partial}{\partial \varphi_{1,2}}\right)$$

and

$$\hat{m}=\begin{pmatrix} m_x & 0 & 0\\ 0 & m_y & 0\\ 0 & 0 & m_\varphi\end{pmatrix}$$

is the mass tensor, where $m_x=m_y=\frac{1}{2}\int|\nabla a^{(0)}|^2\,\mathrm{d}x\,\mathrm{d}y$ and $m_\varphi=\int|a^{(0)}|^2\,\mathrm{d}x\,\mathrm{d}y$, and $\tilde{\phi}$ are the perturbations generated in the interaction. The terms in (26) which do not contain the product of $a_1^{(0)}$ and $a_2^{(0)}$ can be omitted because they either do not contribute to the orthogonality conditions or have the next order of smallness. Besides, it is convenient to rewrite the orthogonality conditions in a more symmetric form. Taking into account that $\nabla_1 a_2^{(0)}\equiv 0$, we obtain

$$\begin{aligned}\mathrm{Re}\int \nabla_1 a_1^{(0)*}\tilde{\phi}\,\mathrm{d}x\,\mathrm{d}y
&=\mathrm{Re}\int \nabla_1(a_1^{(0)}+a_2^{(0)})^*\tilde{\phi}\,\mathrm{d}x\,\mathrm{d}y\\
&=\mathrm{Re}\int \nabla_1(a_1^{(0)}+a_2^{(0)})^*[\beta|a_1^{(0)}+a_2^{(0)}|^2(a_1^{(0)}+a_2^{(0)})\\
&\quad-|a_1^{(0)}+a_2^{(0)}|^4(a_1^{(0)}+a_2^{(0)})]\,\mathrm{d}x\,\mathrm{d}y\\
&=\nabla_1\frac{1}{2}\int\left(\frac{\beta|a_1^{(0)}+a_2^{(0)}|^4}{2}-\frac{|a_1^{(0)}+a_2^{(0)}|^6}{3}\right)\mathrm{d}x\,\mathrm{d}y=\nabla_1 U(r_{12},\varphi_{12}).\end{aligned}$$

The value

$$U(r_{12},\varphi_{12})=\frac{1}{2}\int\left(\frac{\beta|a_1^{(0)}+a_2^{(0)}|^4}{2}-\frac{|a_1^{(0)}+a_2^{(0)}|^6}{3}\right)\mathrm{d}x\,\mathrm{d}y \tag{27}$$

which coincides with a non-square part of the free energy functional can be considered as a pair interaction potential. Then, the equations of spiral motion can be written in a gradient form

$$m_x\dot{x}_{1,2}=-\frac{\partial}{\partial x_{1,2}}U \qquad m_y\dot{y}_{1,2}=-\frac{\partial}{\partial y_{1,2}}U \qquad m_\varphi\dot{\varphi}_{1,2}=-\frac{\partial}{\partial \varphi_{1,2}}U. \tag{28}$$

The expressions for the potential U can be calculated taking into account that the distance between the spirals $R=[(x_1-x_2)^2+(y_1-y_2)^2]^{1/2}$ is large and the field of the second spiral can be replaced by an asymptotic expression for $\rho\to\infty$. Then

$$U\approx\frac{1}{2}\int(\beta|a_1^{(0)}|^2-|a_1^{(0)}|^4)(a_1^{(0)}a_2^{(0)*}+a_1^{(0)*}a_2^{(0)})\,\mathrm{d}x\,\mathrm{d}y$$

$$=\mathrm{Re}\int(\beta|a_1^{(0)}|^2-|a_1^{(0)}|^4)a_1^{(0)}a_2^{(0)*}\,\mathrm{d}x\,\mathrm{d}y. \tag{29}$$

In the framework of the Ginzburg-Landau equation with long-wave instability (15), the asymptotic expression for the field of the second spiral can be written in the form

$$a_2^{(0)}(\rho)\sim\frac{A_0}{\sqrt{\rho_2}}\exp[-\rho_2+\mathrm{i}(m\theta_2-\varphi_2)]$$

$$\approx\frac{A_0\exp[-R+\mathrm{i}(m\psi_{12}-\varphi_2)]}{\sqrt{R}}\exp[r\cos(\theta_1-\psi_{12})]\qquad A_0=\text{constant} \tag{30}$$

where $X=x-x_1$, $Y=y-y_1$, $R=\sqrt{X^2+Y^2}$ are difference coordinates and $\psi_{12}=\tan^{-1}$, (Y/X) is the viewing angle of the second spiral from the site of the first spiral, $r=[(x-x_1)^2+(y-y_1)^2]^{1/2}$.

Substituting (30) into (29) for like spirals ($m=1$) yields

$$U=c\frac{\mathrm{e}^{-R}}{\sqrt{R}}\cos(\varphi_1-\varphi_2) \tag{31}$$

where

$$c=\tfrac{1}{2}A_0\,\mathrm{Re}\int r(\beta\phi^{(0)^3}-\phi^{(0)^5})\exp(-r\cos\xi-\mathrm{i}\xi)\,\mathrm{d}\xi\,\mathrm{d}r\equiv\text{constant}.$$

As a result, we obtain an equation for difference coordinates:

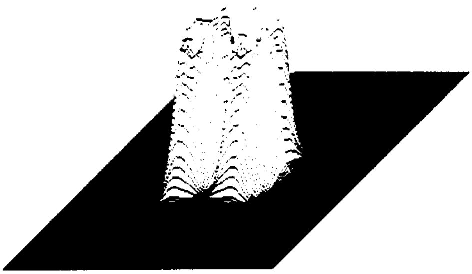
$$\dot{x}=-c_1\cos\varphi\frac{\partial}{\partial x}\left(\frac{\mathrm{e}^{-R}}{\sqrt{R}}\right)$$
$$\dot{y}=-c_1\cos\varphi\frac{\partial}{\partial y}\left(\frac{\mathrm{e}^{-R}}{\sqrt{R}}\right)\qquad\dot{\varphi}=+c_2\sin\varphi\frac{\mathrm{e}^{-R}}{\sqrt{R}} \tag{32}$$

where $c_{1,2}=\text{constant}$ and $\varphi=\varphi_2-\varphi_1$. This system is readily integrated because the value $X/Y=\text{constant}$ is retained. The spirals either collapse or vanish to infinity without bound state formation depending on the sign of the constant c.

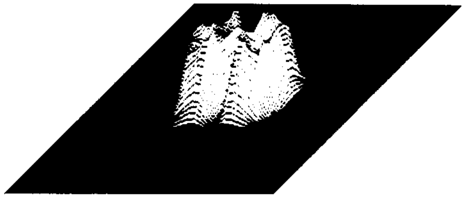
For $m=-1$ (unlike spirals) we obtain the following expression for the potential:

$$U=\frac{c\,\mathrm{e}^{-R}}{\sqrt{R}}\cos(\varphi+2\psi_{12}). \tag{33}$$

Simple verification of the equations generated by this potential shows that it has no fixed points. Hence, in view of the gradient form of the system, unlike spirals do not form bound states either and tend to scatter one another.



(*a*)



(*b*)

Figure 4. Distribution of $|a(x, y)|^2$ for bound states for (16): (*a*) rotating in the same direction ($m = 1$); (*b*) rotating in the opposite directions ($m = -1$).

The situation is essentially different in media with short-wave instability, described by (16). The asymptotic expressions for the spiral tail have the following form:

$$a_2^{(0)} \approx \frac{A_0 \exp[-\alpha R + \mathrm{i}(m\psi_{1,2} - \varphi_2)}{\sqrt{R}} \cos(\tilde{k}R + \xi_0)$$

$$\times \cos(\tilde{k}R + \xi_0) \exp[-\alpha r \cos(\theta - \psi_{12})] \qquad A_0, \xi_0 = \text{constant}$$

where α, $\tilde{k}$ are some constants, characterising oscillatory decay of the spiral 'tail'

$$\alpha = |\mathrm{Re}\sqrt{\mathrm{i} - k_0^2}| \qquad \tilde{k} = |\mathrm{Im}\sqrt{\mathrm{i} - k_0^2}|.$$

Then, for like spirals we shall have

$$U = c\frac{\mathrm{e}^{-\alpha R}}{\sqrt{R}} \cos(\tilde{k}R + \xi_0) \cos\varphi \qquad c = \text{constant}$$

and for unlike spirals we shall have

$$U = c\frac{\mathrm{e}^{-\alpha R}}{\sqrt{R}} \cos(\varphi + 2\psi_{12}) \cos(\tilde{k}R + \xi_0).$$

In either case these potentials have a countable number of stable fixed points corresponding to bound states: spiral dipoles (when $m = -1$) and double spirals (when $m = 1$). These states were observed in a direct numerical experiment with (16). The results are shown in figure 4. Therefore, one may hope that a stable toroidal vortex exists in the framework of (16).

5. Spiral wave motion in media with soft excitation

For such media the GL equation that can be obtained, for example, near a long-wave instability threshold (Malomed and Rudenko 1988), can be written as

$$\frac{\partial a}{\partial t} = a - (1+\mathrm{i}\beta')|a|^2 a + (1-\mathrm{i}c)\Delta a \qquad \beta', c = \text{constant}. \tag{34}$$

This equation has a spiral solution in the form (5) (Hagan 1982, Mikhailov and Krinsky 1983, Malomed 1986) but, in contrast to the solutions considered above, here as $\rho \to \infty$, $\phi^{(0)}(\rho) \to 1$. The stability of such a solution for small β', c was proved in Hagan (1982). For $\beta' = c = 0$ this equation can be written in the gradient form (1) with the 'free energy' functional:

$$F = -\int \left(|a|^2 - \frac{|a|^4}{2} - |\nabla a|^2 \right) \mathrm{d}x\, \mathrm{d}y. \tag{35}$$

Consider now the perturbed equation (34) in the form

$$\frac{\partial a}{\partial t} = a - (1+\mathrm{i}\beta')|a|^2 a + \Delta a + \varepsilon\tilde{\phi}[a]. \tag{36}$$

As in the previous case, we shall seek solutions close to the spiral (5), (6). Then for the functions u_n and v_n we shall have

$$\begin{aligned} &\Delta u_n + (1 - 2\phi^{(0)^2} + \mathrm{Re}(a^{(0)})^2)u_n - \mathrm{Im}(a^{(0)})^2 v_n = H_{\mathrm{R}}^{(n)} \\ &\Delta v_n + (1 - 2\phi^{(0)^2} - \mathrm{Re}(a^{(0)})^2)v_n - \mathrm{Im}(a^{(0)})^2 u_n = H_{\mathrm{I}}^{(n)} \\ &a_n = u_n + \mathrm{i} v_n. \end{aligned} \tag{37}$$

Here all values have the same sense as in (9). Under the boundary conditions for u_n, v_n in the form

$$\left(\frac{\partial u_n}{\partial \rho}\right)^2 + \left(\frac{\partial v_n}{\partial \rho}\right)^2 \to 0$$

when $\rho \to \infty$, $u_n^2(0) + v_n^2(0) < \infty$ the system is self-conjugate as a whole, therefore the solutions of the conjugate system (37) coincide with (10) and the solvability conditions coincide with (11). However, this case differs significantly from the previous one: the functions $a^{(0)}$, $a_x^{(0)}$, and $a_y^{(0)}$ are not quadratically integrable because $|a^{(0)}|^2 \approx 1 - 1/\rho^2$ for $\rho \to \infty$ and the integrals in (11) diverge when $\rho \to \infty$. As a consequence, the solvability conditions are not met for the arbitrary function $\tilde{\phi}[a]$. Physically, this is explained by the fact that the spiral wave field in (34) does not vanish and the wave energy is infinite. Even for weak perturbations, in (34) we must take into account distortions in the wave structure as $\rho \to \infty$, i.e. consider 'radiation fields'. Because the integration is over an infinite surface, the radation field energy can also be infinite and affect the behaviour of the spiral wave significantly†. Therefore, in contrast to localised spiral waves which interact with each other and with radiation by means of 'tails', in this case waves may interact by means of radiation that is difficult to take into account.

We shall now be concerned with a particular class of perturbations. Assume that the supercriticality parameter D^2 in (35) is a slow function of coordinates and time. Besides, we shall take into account weak complexity $\beta' \neq 0$. Consider for definiteness the equation

$$\frac{\partial a}{\partial t} = (D^2(x, y, t) + \mathrm{i}\varepsilon\omega(x, y, t))a - (1+\mathrm{i}\beta')|a|^2 a + \Delta a. \tag{38}$$

† The radiation emerging in the spiral-boundary interaction was taken into account by Biktashev (1989).

To a zeroth approximation with respect to ε the functions D^2 and ω can be assumed constant, which yields a solution of the form

$$a^{(0)}(x, y, t) = \frac{1}{D} \phi^{(0)}(D\rho)\, e^{i(\theta - \varphi_0)}. \tag{39}$$

For $\varepsilon \neq 0$ we shall seek a solution in the form (8). The divergence in (11) will be eliminated as follows. We shall introduce a characteristic size of the spiral wave: $r_c \sim 1$ is the core radius. In addition, by analogy with Hagan (1982) we shall introduce the size of the 'internal asymptotic expansion' R_i, which meets the condition $r_c \ll R_i \ll 1/\varepsilon$. Assume that in the limit

$$\varepsilon = \|\nabla D\| \to 0$$

the spiral behaves as an entity and the solution of the form (39) is retained for $\varepsilon \neq 0$. To the first approximation, we shall not consider the field behaviour in the external region $\rho \gg R_i$. Assume also that the scale of the variation of $\omega(x, y, t)$ and $D^2(x, y, t)$ is much larger than R_i (and not r_c as for the case of localised spirals).

In this case the correction $H^{(1)}$ has the form

$$H^{(1)} = i\dot{\varphi}_0 a^{(0)} + \dot{x}_0 \frac{\partial a^{(0)}}{\partial x} + \dot{y}_0 \frac{\partial a^{(0)}}{\partial y} - i\beta' |a^{(0)}|^2 a^{(0)}$$
$$+ i\omega(x, y, t) a^{(0)} - 2\left(\frac{\partial^2 a^{(0)}}{\partial x \partial \xi} + \frac{\partial^2 a^{(0)}}{\partial y \partial \eta}\right) \tag{40}$$

where

$$(\xi, \eta) = (\varepsilon x, \varepsilon y).$$

Substituting (40) into (11) and choosing appropriate $\dot{x}_0$, $\dot{y}_0$ and $\dot{\varphi}_0$ we shall eliminate the terms growing under the orthogonality conditions as $\rho \to \infty$. The resulting expressions (11) will become finite and might be easily compensated in the following approximation by the addition of a small perturbation to u_1, v_1, which can be interpreted as the appearance of weak radiation (see also Biktashev (1989)).

Analysis shows that the following equations must be fulfilled:

$$\dot{\varphi}_0 = \beta' - \omega(x_0, y_0, t) \qquad \dot{x}_0 = \frac{1}{D^2} \frac{\partial}{\partial x_0} D^2(x_0, y_0, t) \qquad \dot{y}_0 = \frac{1}{D^2} \frac{\partial}{\partial y_0} D^2(x_0, y_0, t). \tag{41}$$

Specifying the form of D^2 and ω we can find a complex, even stochastic, spiral wave drift over the x, y plane.

However, this method does not hold for the analysis of the effect of perturbations of a more general form, for example, the perturbations caused by the interaction of spirals in a pair. Therefore, for interactions of such a type we shall use a different approach.

6. Drift of a spiral pair. Analogy with disclination drift in liquid crystals.

Rewrite (34) (for $\beta' = c = 0$) in the variables of the modulus $I = |a|^2$ and phase $\phi = \arg a$.

Then we shall obtain a system of the form

$$\frac{\partial I}{\partial t} = I - I^3 + \Delta I - (\nabla\phi)^2 I$$
$$\frac{\partial \phi}{\partial t} = \Delta\phi + 2\left(\frac{\nabla I}{I}, \nabla\phi\right). \tag{42}$$

We can easily make sure that in the spiral solution (5), $\nabla I \perp \nabla\phi$, and the phase distribution is described by the equation

$$\frac{\partial \phi}{\partial t} = \Delta\phi. \tag{43}$$

This equation can also be written in the gradient form (1) with the free energy functional

$$F = \frac{1}{2}\int |\nabla\phi|^2 \, dx \, dy \tag{44}$$

This is precisely the equation which describes the director field distribution in nematic crystals (see, for example, Chandrasekhar and Ranganath 1986). Thus, we can use the spiral wave-disclination analogy and the results known for it. Equation (43) has the known solutions in the form of individual disclination of strength S (Chandrasekhar and Ranganath 1986):

$$\phi(\rho, \theta) = S\theta + C \tag{45}$$

where C = constant and $S = \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}\ldots$. However, for the spiral waves of interest there must be no disclinations of half-integer strength, since the directions of ϕ and $-\phi$ are significant in (34). Therefore in our case individual disclinations (corresponding to single-armed spiral waves which are stable in (34)) have the strength $|S| = 1$. Here, in contrast to liquid crystals, only the modulus of S is of interest (the sign of S denotes the direction of spiral rotation).

We shall use the spiral wave-disclination analogy for the investigation of spiral wave interaction. It is known that in an unbounded medium a single disclination (and, hence, a spiral) has infinite energy. To eliminate this disagreement, following Chandrasekhar and Ranganath (1986), we shall assume that the region R_b where the spiral is located is bounded though very large in comparison with the size of the core: $R_b \gg r_c$. Besides, the disclination energy diverges as $\rho \to 0$. In the theory of liquid crystals this singularity is omitted because when $\rho \to 0$ the director field structure is different: as $\rho \to 0$ the nematic transforms to an ordinary liquid and the energy in the region $0 < \rho < r_c$ is finite and equal to E_c.

The same approach holds for spiral waves. However, the finiteness of the core energy has a different origin. In the initial equation (35) the value $\nabla\phi$ enters in the expression for free energy only in the combination $I^2|\nabla\phi|^2$. Because when $\rho \to 0$, $I \sim \mu\rho$ and μ = constant, $\int_0^{r_c} I^2|\nabla\phi|^2 \, dx \, dy = E_c$ is a finite value. This circumstance enables us to consider the energy of the spiral wave core also to be finite. Under these assumptions the spiral wave energy will be finite and equal to

$$E = \frac{1}{2}\int |\nabla\phi|^2 \, dx \, dy = E_c + \pi \ln \frac{R_b}{r_c}. \tag{46}$$

Now we consider the interaction of two spiral waves. Unlike in the case considered in section 4, the solution in the form of two spiral waves is not a superposition of solutions of the form (24) because here $|a^{(0)}|^2 \to 1$ as $\rho \to \infty$.

A two-spiral solution can be constructed as follows. Far from each spiral, the field must be unperturbed, i.e., $I=|a^{(0)}|^2\to 1$, provided that the distance between the spiral cores is much larger than the core radius r_c. In the vicinity of each of the spirals the field must be close to the self-field of the spiral, i.e. it is described by (5). Therefore, everywhere except small regions near the spiral cores $I\sim 1$ the term $(\nabla I \nabla \phi)$ in (42) can be omitted and the equation for intensity will be separated, while the phase of the two-spiral solution will again be described by (43)†. Equation (43) is linear; therefore, sufficiently far from the spiral cores, the field will be a superposition of the fields of individual spirals. Then the solution for ϕ may be represented as (Chandrasekhar and Ranganath 1986)

$$\phi(x,y,t)=S_1\tan^{-1}\left(\frac{y-y_1}{x-x_1}\right)+S_2\tan^{-1}\left(\frac{y-y_2}{x-x_2}\right)\qquad |S_{1,2}|=1. \tag{47}$$

Below we shall assume that the distance between the spiral cores meets the relation $r_c \ll R_{12}=[(x_1-x_2)^2+(y_1-y_2)^2]^{1/2}\ll R_b$. We shall use (47) for the calculation of the energy of spiral interaction. Substituting (47) into (44), after integration we shall obtain a known expression for the energy of disclination pairs (see Chandrasekhar and Ranganath 1986)

$$E=2\pi(S_1+S_2)\ln\frac{R_b}{r_c}+E_{12}+2E_c \tag{48}$$

where

$$E_{12}=-S_1S_2\ln(R_{12}/r_c) \tag{49}$$

is the interaction energy. The force of spiral interaction is determined by $\boldsymbol{F}_{12}=-\nabla E_{12}$. From this expression it follows that the spirals rotating in the same direction $(S_1=S_2)$ repel and those rotating in opposite directions attract each other. It follows from (43) and (49) that under the action of forces the spirals will move as if in a viscous medium with a definite velocity. The 'frictional force' that will appear in this case will balance the force of interaction. From the condition of the equality of the frictional force and the force of spiral pair interaction, we can determine the law according to which $x_{1,2}$ and $y_{1,2}$ are time dependent (to the first approximation acceleration can be neglected). To this end, consider a spiral moving with a constant (but low) velocity v along, for example, x. For $|v|\ll 1$ the solution can be represented in the form

$$\phi(x,y,t)\approx\tan^{-1}\left(\frac{y}{x-vt}\right). \tag{50}$$

Substituting (50) into (44), we shall obtain the relation

$$v\phi_x=\Delta\phi. \tag{51}$$

Under the action of the frictional force the energy in the system will change. Since the energy variation is equal to the work done by the frictional force F_v, the displacement of the spiral within $\mathrm{d}l$ in time $\mathrm{d}t$ from the conservation-of-energy law yields the energy variation

$$\mathrm{d}l\,F_v=-\mathrm{d}E=-(F(t+\mathrm{d}t)-F(t)) \tag{52}$$

† Such an approach does not hold for the descriptions of the interaction of the spiral waves considered in sections 3 and 4. In this solution $|a^{(0)}|^2\to 0$ as $\rho\to\infty$, therefore the term $(\nabla I/I)\nabla\phi$ cannot be omitted because small changes in intensity lead to large changes in phase.

where F_v is the frictional force. Taking into account that $v = \mathrm{d}l/\mathrm{d}t$ and using (51), we shall have

$$vF_v = -\frac{\partial F}{\partial t} = \frac{1}{2}\int \frac{\partial}{\partial t}|\nabla\phi|^2\,\mathrm{d}x\,\mathrm{d}y \approx -v^2\int \phi_x^2\,\mathrm{d}x\,\mathrm{d}y$$
$$= -\frac{1}{2}v^2\int |\nabla\phi|^2\,\mathrm{d}x\,\mathrm{d}y$$

Then the frictional force will be

$$F_v = -\frac{v}{2}\int |\nabla\phi|^2\,\mathrm{d}x\,\mathrm{d}y = -vF_0 \tag{53}$$

where F_0 is the intrinsic free energy of an isolated spiral. Equating the frictional force to the force of interaction, we shall obtain the following equations of motion for the spirals rotating in opposite directions:

$$\begin{aligned} F_0\dot{x}_{1,2} &= \frac{\partial}{\partial x_{1,2}} E_{1,2} = \frac{x_{2,1} - x_{1,2}}{R_{12}^2} \\ F_0\dot{y}_{1,2} &= \frac{\partial}{\partial y_{1,2}} E_{1,2} = \frac{y_{2,1} - y_{1,2}}{R_{12}^2}. \end{aligned} \tag{54}$$

Or, introducing R_{12}, we shall obtain

$$F_0\dot{R}_{12} = -1/R_{12}. \tag{55}$$

Thus, as $t \to \infty$ this spiral pair collapses. However, the collapse time depends on F_0, i.e. on the medium size. In an unbounded medium $F_0 \to \infty$ and, hence, $|\boldsymbol{v}| \to 0$, i.e. the spiral waves 'freeze in'.

A similar analysis can be performed for spiral pair interaction in terms of complete equations (34) or (42). For example, we can calculate the frictional forces acting on the spiral drifting along x, y and rotating around θ. Analogous considerations for energy variation give the friction coefficients along x

$$m_x = \int |a_x^{(0)}|^2\,\mathrm{d}x\,\mathrm{d}y = \frac{1}{2}\int |\nabla a^{(0)}|^2\,\mathrm{d}x\,\mathrm{d}y$$

along y

$$m_y = \int |a_y^{(0)}|^2\,\mathrm{d}x\,\mathrm{d}y = \frac{1}{2}\int |\nabla a^{(0)}|^2\,\mathrm{d}x\,\mathrm{d}y$$

and along θ

$$m_\theta = \int |a^{(0)}|^2\,\mathrm{d}x\,\mathrm{d}y.$$

Comparison with (14) shows that these coefficients coincide with those for $\dot{x}_0$, $\dot{y}_0$, and $\dot{\varphi}_0$ that were derived asymptotically in section 2. For the spiral to drift in an unbounded medium all forces must be proportional to the corresponding damping coefficients m_x, m_y and m_θ, i.e. they also must be infinite (this case was considered in section 5). If the forces are finite, then the spirals are frozen, as we have already mentioned, in an unbounded medium and perturbations distort the spiral field but do not shift the spiral (it is natural to interpret these perturbations as radiation).

A complete functional (35) can be used for the calculation of the coupling potential for the two waves in terms of the initial equation (36). Apparently, the case considered here corresponds to the allowance made only for the last term in (35) because $|\nabla a|^2 \sim |\nabla \phi|^2$, as $\rho \to \infty$.

Allowance made for the first two terms adds to the logarithmic coupling potential (49) the power terms $R_{12}^{-\gamma}$, $\gamma > 1$ that are negligibly small when $R_{12} \to \infty$. Thus, in the 'phase approximation' used in section 6 we find the main expansion term of the coupling potential of spiral pairs.

To conclude this section we would like to note that the approach developed may also be used for describing the behaviour of non-localised spiral waves in media with hard excitation. It is known that besides the solutions localised when $\rho \to \infty$, in such media there exist stable non-localised solutions: $|a|^2 \to$ constant when $\rho \to \infty$. Because only the asymptotic behaviour for $\rho \to \infty$ is essential in the analysis performed here, all considerations in section 6 are valid (with the exception, most probably, of the core energy E_c).

7. Discussion

In conclusion we shall discuss the problems related to the results obtained in this work: (i) the methods for describing solutions in the form of quasiharmonic spiral waves and (ii) the origin of spatial disorder in nonlinear non-equilibrium media.

(i) It is essential that quite different mechanisms may be responsible for the formation and stable existence of spiral waves (waves with rotating wavefronts) which look very much alike. Like plane waves that may be gradient in nature, quasiharmonic or resemble shock waves, depending on the properties of the medium (the presence of excitation threshold, dispersion, the type of nonlinearity), spiral waves may also be different, depending on the medium properties. Therefore different methods are needed for their description.

Today spiral waves in excitable media, i.e. media with relaxation point dynamics (for example, when an isolated element of the medium is described by a trigger or a multivibrator equation), seem to be best investigated. Usually, when the waves propagate in such a medium (like a cardiac muscle or a nerve fibre (Krinsky and Yakhno 1980, Kuramoto 1982)) the effects of long-range interaction are not essential and spiral waves may be described in a kinematic approximation (Brazhnik *et al* 1988, Meron and Pelce 1988)) that is based on the consideration of waves which are nearly kinematic. A pure case of kinematic waves are gradient waves in a strip of falling dominoes. It is natural that the asymptotic method developed above does not hold for the analysis of waves which are nearly kinematic.

On the other hand, the kinematic approach will not, evidently, hold for the investigation of another limiting class of spiral waves—quasiharmonic dispersion waves. Such waves are typical, in particular, of some two-dimensional hydrodynamic flows (Huerre 1987), and may be generated in two-dimensional reactors where an autocatalytic chemical reaction proceeds (Kuramoto 1982) and so on. The method proposed in this paper is suitable for these very waves.

(ii) We have not obtained rigorous results on spatio-temporal chaos and random walks of localised structures in two-dimensional media (it is very hard to provide such long-time numerical simulations), but the one-dimensional case was investigated in our recent paper (Aranson *et al* 1989). We hope the results of this paper are sufficiently common.

Note in conclusion that it is interesting to generalise the problems under study to a three-dimensional case. Here it is natural to use as the generating solutions gradient or nearly gradient models demonstrating stable localised particle-like solutions. A new class of such solutions was found, in particular, by Gorshkov *et al* (1989).

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